NCRA VII 2021-Online Conference Dedicated to Tariq Rizvi-
5-7th July 2021

Properties of Strongly Harmonic and Gelfand modules: idioms, frames and associated topological spaces.

Martha Lizbeth Shaid Sandoval-Miranda,
Universidad Autónoma Metropolitana,
Campus Iztapalapa, Mexico City
marlisha@xanum.uam.mx

a joint work with
Mauricio Medina, BUAP; **Ángel Zaldivar**, CUCEI UdeG; **Lorena Morales**, UNAM

1 Introduction
- Strongly harmonic and Gelfand rings
- Demarco-Orsatti-Simmons Theorem

2 What about these ideas for the case of modules?
- Idioms, quantales and nuclei
- Normal idioms

3 Applications to modules
- Prime Spectrum of a Module
- Strongly Harmonic Modules
- Gelfand Modules

4 References

Introduction

- In the study of rings and their module categories, we have used techniques and theory of lattices.
    - lattice of preradicals, lattice of classes of modules, lattice of hereditary torsion theories, linear filters,, lattices of submodules, etc.
    - M. J. Arroyo, R. Fernández-Alonso, F. Raggi, H. A. Rincón, J. Ríos,, C. Signoret,..
- The theory of frames (local, complete Heyting algebras) arises in the study of topological spaces by means of the frame of open sets (it point-free topology). This involves the study of interactions between topological and algebraic concepts by means of the frame of open sets of a space.
    - J. Picado, A. Pultr (*Frames and Locales, 2012*)
    - P. T. Johnstone (*Stone Spaces, 1986*)

- Harold Simmons, carried out research in the theory of *idioms* (complete, modular, and upper-continuous lattices), and the study of situations arising in ring theory, in combination with techniques from point-free topology.
  - H. Simmons, Some methods of attaching a topological space to a ring, Private communication, 2016.
- In collaboration with Mauricio Medina Bárcenas, Lorena Morales Callejas, and Angel Zaldivar Corichi, we follow the Simmons research line for the study of modules, particularly in the category $\sigma[M]$. We investigate certain frames associated to modules; and in particular those that turn out to be spatial (i.e., isomorphic to the frame of open sets of a topological space).

Strongly harmonic and Gelfand rings

Demarco-Orsatti-Simmons Theorem

Definition

A ring R is said to be strongly harmonic if it satisfies the following condition: for each pair of distinct maximal ideals M_1 and M_2, there are ideals I_1, I_2 such that $I_1 \nsubseteq M_1$, $I_2 \nsubseteq M_2$ and $I_1 I_2 = 0$.

Definition

A ring R is said to be left Gelfand (Gelfand ring) if it satisfies the following condition: for each pair of distinct maximal ideals M_1 and M_2, there exist left ideals I_1, I_2 such that $I_1 \nsubseteq M_1$, $I_2 \nsubseteq M_2$ and $I_1 I_2 = 0$. It can be proved that the condition is symmetric.

- The general definition of a strongly harmonic ring was introduced by K. Koh in 1972. [1]. In this paper, it was shown that the space of maximal ideals $Max(R)$ of a strongly harmonic ring R with hull-kernel topology is a compact Hausdorff space.
- Later, in 1979 C. J. Mulvey [2] introduced the Gelfand rings.
  It was proved that for these rings, the space of maximal ideals is also compact Hausdorff (it is proved that any Gelfand ring is strongly harmonic).

[1] [Koh72] On a representation of a strongly harmonic ring by sheaves. K. Koh. Pacific Journal of Mathematics, vol. 41-2, pages 459–46, 1972. Mathematical Sciences Publishers

[2] [Mul79] A generalization of Gelfand duality. Mulvey, C.J. Journal of Algebra, vol. 56-2, pages 499–505, 1979. Academic Press.

- In this paper, it was shown that the space of maximal ideals $Max(R)$ of a strongly harmonic ring R with hull-kernel topology is a compact Hausdorff space. [Koh72,Theorem 3.7]
- Borceux, F. and van den Bossche, G. introduced a representation for rings based on the frame $\Psi(R)$ defined as the set of *pure ideales*. [3]It can be proved that the frame $\Psi(R)$ works as a good space for unifying known representations; and it is shown that for Gelfand rings, the point space $\Psi(R)$ is homeomorphic to $Max(R)$ with the hull-kernel topology, equivalently,

$$\Psi(R) \cong \mathcal{O}(Max(R))$$

. In particular, this means that $\Psi(R)$ is a spatial frame. [4] [5]

[3] [BvdB06] Algebra in a localic topos with applications to ring theory. Borceux, F. and van den Bossche, G. Vol. 1038. Springer, 2006.

[4] [BSvdB84] A sheaf representation for modules with applications to Gelfand rings.Borceux, F. and Simmons, H. and van den Bossche, G. Proceedings of the London Mathematical Society, vol. 3-2, 230–246, 1984. Wiley Online Library

[5] Sheaf representations of strongly harmonic rings. H. Simmons. Proceedings of the Royal Society of Edinburgh Section A: Mathematics. Vol. 99, 3-4,pages 249–268, 1985. Royal Society of Edinburgh Scotland Foundation

- H. Simmons studied properties of strongly harmonic rings, applying theory of point-free topology and idioms. [6]
- In a joint work [7] with M. Mediana, L. Morales y A. Zaldivar, we introduced a notion of *strongly harmonic module* and *Gelfand module*. In addition, we explored properties of these modules, following those already known in the case of rings. We study the space of fully invariant maximal submodules $Max^{fi}(M)$ for strongly harmonic modules and $Max(M)$ for Gelfand modules; and we relate both spaces to the point space of $\Psi(M)$. (See also [MSZ21][8])

6 [Sim]Some methods of attaching a topological space to a ring. Simmons, H. Notes in private communication with A. Zaldivar-Corichi.

7 [MMSZ20]On strongly harmonic and Gelfand modules. M. Medina Barcenas, L. Morales Callejas, M. L. S. Sandoval Miranda, A. Zaldivar. Communications in Algebra, Volume 48-5. 2020.

8 [MMSZ21] On the De Morgan's laws for modules. Medina Barcenas, M. L. S. Sandoval Miranda, A. Zaldivar.(Preprint 2020) https://arxiv.org/abs/2003.05607

Theorem (Demarco-Orsatti[9]-Simmons[10])

For a commutative ring R the following conditions are equivalent:

- *R es a pm-ring (each prime ideal is contained in a unique maximal ideal)*
- *$Max(R)$ is a retract of $Spec(R)$*
- *$Spec(R)$ is normal*
- *R satisfies the following condition: for every pair of distinct maximal ideals M_1 and M_2 there exist ideals I_1, I_2 such that $I_1 \nsubseteq M_1$, $I_2 \nsubseteq M_2$ and $I_1 I_2 = 0$.*

[9] G. Demarco and A. Orsatti, Commutative rings in which every prime idcal contained in a unique maximal ideal. Proc. Amer. Math. Sot. 30 (1971) 4599466.

[10] H. Simmons, Reticulated rings, J. Algebra 66 (1980) 169-192; Errata, ibid., 74 (1982) 292.

What about these ideas for the case of modules?

Notation:

- $R-Mod$ Category of left $R-$modules.
- $\Lambda(M)$ the lattices of submodules of a given module M. Actually, it is an idiom.
- $\sigma[M]$ is the Grothendieck abelian category whose objects are the R-modules N which are $M-$subgenerated (i.e. there exist morphisms $0 \to N \overset{\alpha}{\to} Q$ and $M^{(X)} \overset{\beta}{\to} Q \to 0$)
- $\Lambda^{fi}(M) = \{N \in \Lambda(M) \mid N$ is a fully invariant submodule of $M\}$, the lattice of fully invariant submodules of M. [11]

[11] Let N be a submodule of M. It is said that N is a fully invariant submodule of M if it satisfies that for every endomorphism, $f : M \to M$, $f(N) \subseteq N$.

Some lattice concepts

Definition

An **idiom** $(A, \leq, \bigvee, \wedge, 1, 0)$ *is a complete,* [12] *modular and upper continuous lattice. That is,* A *is a complete lattice satisfying the following distributive laws:*

$$a \wedge (\bigvee X) = \bigvee \{a \wedge x \mid x \in X\},$$

for each $a \in A$ *and* $X \subseteq A$ *is directed; and*

$$a \leq b \Rightarrow (a \vee c) \wedge b = a \vee (c \wedge b)$$

for each $a, b, c \in A$*.*

[12] We say that a partially ordered set $(A, \preceq)$ (i.e. $\preceq$ is r.a.t.) is a **lattice,** if it satisfies that for every $a, b \in A$, there exist $a \vee b \in A$ and $a \wedge b \in A$. In the case that for all $X \subseteq A$ it is satisfied that there are $\bigvee X, \bigwedge X \in A$, A is a **complete lattice.**

Definition

Let A be a complete lattice.

(a) *A is* **distributive** *if it is satisfied that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, for each $a \in A$ and $b, c \in A$.*

(b) *A is a* **frame** *if it is satisfied that $a \wedge \bigvee X = \bigvee \{a \wedge x \mid x \in X\}$, for each $a \in A$ and $X \subseteq A$,*

Example

Let us consider X a topological space and $\mathcal{O}(X) = \{U \subseteq X \mid U \text{ is open}\}$. Then, $(\mathcal{O}(X), \subseteq)$ is a frame, where

$$\bigvee_{i \in I} \{U_i\} = \bigcup_{i \in I} \{U_i\} \text{ and } \bigwedge_{i \in I} \{U_i\} = Int(\bigcap_{i \in I} \{U_i\}).$$

Quantales and quasi-quantales

Definition

Let A be a complete lattice and $*: A \times A \to A$ an associative binary operation. It is said that A is a quantale[13] *(quasi-quantale)*[14]*, if*

$$a * (\bigvee X) = \bigvee_{i \in I} \{a * x_i \mid x_i \in X\} \text{ y}$$

$$(\bigvee X) * b = \bigvee_{i \in I} \{x_i * b \mid x_i \in X\}$$

for each $a \in A$ and each (directed) subset X of A.

13 The concept of quantale arose around 1920, when W. Krull, followed by R. P. Dilworth and M. Ward, considered a lattice of ideals equipped with multiplication. The "quantale" term is due to C. J. Mulvey.

14 The quasi-quantale definicition was introduced in [[MSZ15]] by M. Medina, M.L.S. Sandoval y A. Zaldivar.

Example

(a) *Every frame is a quantale, in this case, the operación* $\star$ *is* $\wedge$.

(b) *In particular, the frame of open sets in a topological space.*

(c) *Give a ring* R, $\Lambda^{fi}(R) = Id(R)$ *is a quantale, with the usual multiplication of ideals.*

(d) *(The power set of a semigroup) Let* $(S, \cdot)$ *be a semigroup and* $\mathcal{P}(S)$ *the set of its subsets. Then,* $\mathcal{P}(S)$ *is a complete lattice, and a multiplicacion in* $\mathcal{P}(S)$ *can be defined as follows:*

$$UV = \{u \cdot v \mid u \in U, v \in V\}$$

for each $U, V \in \mathcal{P}(S)$. *The quantale* $\mathcal{P}(S)$ *is commutative (unitary) if and only if* S *is commutative (a monoid).*

Definition

Let $M \in \mathrm{R\text{-}Mod}$ and $K, L \in \Lambda(M)$. The product of K with L in M is defined as follows:

$$K_M L := \sum\{f(K) \mid f \in Hom_R(M, L)\}.$$

In particular, for $I, J \in \Lambda(R)$, notice that $I_R J = IJ$.
Also, notice that for each $K \in \Lambda^{fi}(M)$, $K_M M = K$.

Proposition

If M is projective in $\sigma[M]$, then:

(a) the product $-_M- : \Lambda(M) \times \Lambda(M) \to \Lambda(M)$ is associative (Beachy, 2002), and

(b) $(\Lambda(M), -_M-)$ is a quasi-quantale.

(c) $(\Lambda^{fi}(M), -_M - |_{fi})$ is a subquasi-quantale of $(\Lambda(M), -_M-)$ right unitary (in this case, M.)

Proposition (CRT18)

If M is a multiplication[15] module over a commutative ring, then the product $-_M-$ is associative.

[15] A module M is *multiplicative* if for every $N \in \Lambda(M)$, there is an ideal I of R such that $N = IM$.

Proposition

Given any morphism of $\bigvee$-semilattices, $f^*\colon A \to B$, there exists a map $f_*\colon B \to A$ such that

$$f^*(a) \leq b \Leftrightarrow a \leq f_*(b),$$

for each $a \in A$ and $b \in B$. That is, f^* and f_* form an adjunction

$$A \underset{f_*}{\overset{f^*}{\rightleftarrows}} B$$

In fact, $f_*(b) = \bigvee\{x \in A \mid f^*(x) \leq b\},$ for each $b \in B$.

This is a particular case of the General Adjoint Functor Theorem.

Definition

Let A be an idiom. A nucleus on A is a monotone function $j\colon A \to A$ such that:

(a) $a \leq j(a)$ *for each* $a \in A$.

(b) j *is idempotent.*

(c) j *is a prenucleus, that is,* $j(a \wedge b) = j(a) \wedge j(b)$.

Given a nucleus j, the set of all *fixed points* of j is denoted by

$$A_j = \{a \in A \mid j(a) = a\}.$$

Definition ([MSZ15])

Let B be a subquasi-quantale of a quasi-quantale A. An element $1 \neq p \in A$ is a prime element relative to B if whenever $ab \leq p$ with $a, b \in B$ then $a \leq p$ or $b \leq p$.

We define the *spectrum of A relative to B* as

$$Spec_B(A) = \{p \in A \mid p \text{ is prime relative to } B\}.$$

In the case $A = B$ this is the usual definition of prime element. We denote the set of prime elements of A by $Spec(A)$.

Proposition ([MSZ15])

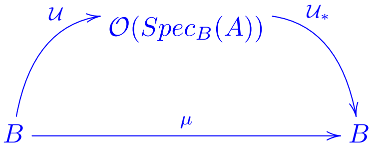
Let B be a subquasi-quantale satisfying $(\star)$ of a quasi-quantale A. Then $Spec_B(A)$ is a topological space with closed subsets given by
$\mathcal{V}(b) = \{p \in_B (A) \mid b \leq p\}$ *with* $b \in B$.
In dual form, the open subsets are of the form
$\mathcal{U}(b) = \{p \in {}_B(A) \mid b \not\leq p\}$ *with* $b \in B$.

Let $\mathcal{O}(Spec_B(A))$ be the frame of open subsets of $Spec_B(A)$. We have an adjunction of $\bigvee$-morphisms

$$B \underset{\mathcal{U}_*}{\overset{\mathcal{U}}{\rightleftarrows}} \mathcal{O}(Spec_B(A))$$

$$\mathcal{U}(b) := U(b)$$

$$\mathcal{U}_*(W) := \bigvee\{b \in B \mid U(b) \subseteq W\}$$



The composite $\mu := U_* \circ U : B \to B$ is a *multiplicative nucleus* , i.e.:

(a $b \leq b'$ implies $\mu(b) \leq \mu(b')$

(b $b \leq \mu(b)$ for each $b \in B$

(c μ is idempotent

(d $\mu(bc) = \mu(b \wedge c) = \mu(b) \wedge \mu(c)$.)

Theorem ([MSZ15])

Let B be a subquasi-quantale satisfying $(\star)$ of a quasi-quantale A and $\mu = \mathcal{U}_* \circ \mathcal{U} \colon B \to B$ as above. Then, the following conditions hold.

(a) *For each $b \in B$, $\mu(b)$ is the largest element of B such that*

$$\mu(b) \leq \bigwedge\{p \in Spec_B(A) \mid p \in \mathcal{V}(b)\}.$$

(b) *μ is a multiplicative nucleus.*

(c) *B_μ is a meet-continuous lattice.*

(d) *If B satisfies that for any $X \subseteq B$ and $a \in B$*

$$(\bigvee X)\, a = \bigvee\{xa \mid x \in X\}$$

then, B_μ is a frame.

Definition

Let A be a multiplicative idiom. We say that A is normal if for every $a, b \in A$ with $a \vee b = 1$, there exist $a', b' \in A$ such that $a \vee b' = 1 = a' \vee b$ and $a'b' = 0$.

Proposition ([MMSZ20])

Let A be a quasi-quantale satisfying $(\star)$[16]. Let μ be the multiplicative nucleus given by the adjoint situation. Then, the following conditions are equivalent

(a) $Spec(A)$ *is a normal space.* [17].

(b) A_μ *is a normal lattice.*

[16] Given a subquasi-quantal B of a quasi-quantale A, we will say B satisfies the condition $(\star)$ if $0, 1 \in B$ and $1b, b1 \leq b$ for all $b \in B$.

[17] For any two disjoint closed F_1y F_2, there exist two open sets U y V,also disjoint, such that $F_1 \subseteq U$ and $F_2 \subseteq V$, respectively

Let B be a subquasi-quantale satisfying $(\star)$ of a quasi-quantale A.
Let S be a subspace of un $Spec_B(A)$, we have the hull-kernel adjunction

$$B \underset{m_*}{\overset{m}{\rightleftarrows}} \mathcal{O}(S)$$

with $m(b) = \mathcal{U}(b) \cap S$. Then $\tau := m_* \circ m \colon B \to B$ is a multiplicative nucleus as in the case of μ.

Theorem ([MMSZ20])

Let B be a subquasi-quantale satisfying $(\star)$ of a quasi-quantale A and let S be a subspace of $Spec_B(A)$. Let τ be the multiplicative nucleus given previously. Then, the following conditions are equivalent.

(a) *S is a normal topological space.*

(b) *B_τ is a normal lattice.*

Applications to modules

Prime Spectrum of a Module

Definition

Let $M \in \mathrm{R\text{-}Mod}$ *and* $M \neq P \in \Lambda^{fi}(M)$. *We say* P *is a prime submodule in* M *if for any fully invariant submodules* K, L *of* M *such that* $K_M L \leq P$, *then* $K \leq P$ *o* $L \leq P$.

Prime and coprime modules
Bican, L., Jambor P., Kepka T., Němec P.
Fundamenta Mathematicae, 107:33-44 (1980).

Applying the theory to the module case

Consider M projective in $\sigma[M]$, such that $\Lambda^{fi}(M)$ is a subquasiquantale of $\Lambda(M)$.

- $LgSpec(M) := Spec_{\Lambda^{fi}(M)}(\Lambda(M))$
  denote $P \in \Lambda(M)$ which are relative prime in $\Lambda^{fi}(M)$; and we called it *Prime spectrum of M.*

- $Spec(\Lambda^{fi}(M)) := Spec_{\Lambda^{fi}(M)}(\Lambda^{fi}(M))$

Remark

To be sure that these spectra are not empty, we can assume that M is coatomic.[18]

[18] A module M is said to be coatómic if it satisfies that for every $N \in \Lambda(M)$, there is $\mathcal{M}$ a maximal submodule of M such that $N \leq \mathcal{M}$.

Theorem ([MSZ15])

Let M be projective in $\sigma[M]$ and coatomic. Entonces:

(a) $LgSpec(M)$ *is a topological space, where the closed sets are given by* $V(L) = \{P \in LgSpec(M) \mid L \leq P\}$, *with* $L \in \Lambda(M)$

(b) $Spec(\Lambda^{fi}(M))$ *is a dese subspace of* $LgSpec(M)$.

(c) *There is a multiplicative nucleus:* $\mu : \Lambda^{fi}(M) \to \Lambda^{fi}(M)$ *satisfying that:*

  1. $\mu(N)$ *is the largest fully invariant submodule of* M *contained in* $\bigcap_{P \in V(N)} P$
  2. $\mu(N) = N$ *if and only if* N *is semiprime*[19] *in* M, *or* $N = M$.

(d) $(\Lambda^{fi}(M))_\mu = SP(M)$ *is a frame, where* $SP(M) := \{N \in \Lambda^{fi}(M) \mid N$ *is semiprime*$\} \cup \{M\}$. *Moreover,* $SP(M) \cong \mathcal{O}(LgSpec(M))$ *are canonically isomorphic as frames.*

---

[19] We say $N \in \Lambda^{fi}(M)$ is *semiprime en* M if whenever $K_M K \leq N$, with $K \in \Lambda^{fi}(M)$, then $K \leq N$.

Corollary

Let M be projective in $\sigma[M]$. The following conditions are equivalent:

(a) $Spec(M)$ *is a normal space.*

(b) *The frame* $SP(M) \cong \mathcal{O}(LgSpec(M))$ *is normal.*

Corollary

The following conditions are equivalent for a ring R:

(a) $Spec(R)$ *is a normal space.*

(b) *The frame* $SP(R)$ *is normal.*

Strongly Harmonic Modules

Definition ([MMSZ20])

A module M is strongly harmonic if for every distinct elements $N, L \in Max^{fi}(M)$ there exist $N', L' \in \Lambda^{fi}(M)$ such that $L' \nleq L$, $N' \nleq N$ and $L'_M N' = 0$.

Remark ([MMSZ20][MSZ20])

1. *Let M be a quasi-projective module and $N \in \Lambda^{fi}(M)$. Then, $N \in Max^{fi}(M)$ if and only if M/N is FI-simple.*
2. *Let M be a quasi-projective strongly harmonic module. Then $M^{(I)}$ is a strongly harmonic module for every index set I.*
3. *Let M be a quasi-projective module. Suppose $M = \bigoplus_I M_i$ is a direct sum with $M_i \in \Lambda^{fi}(M)$. Then M is a strongly harmonic module if and only if M_i is strongly harmonic.*
4. *Let M be a self-progenerator in $\sigma[M]$. Suppose M is strongly harmonic such that $\Lambda^{fi}(M)$ is compact. If M is semiprime and satisfies DML then $Max^{fi}(M)$ is extremally disconnected space.*

Proposition ([MMSZ20])

Let M be a self-progenerator in $\sigma[M]$. Assume M is strongly harmonic such that $\Lambda^{fi}(M)$ is compact. Then $\Psi(M) \cong \mathcal{O}(Max^{fi}(M))$, where

$$\Psi(M) = \{N \in \Lambda^{fi}(M) \mid \forall n \in N, [N + \mathrm{Ann}_M(Rn) = M]\}^{20}.$$

[20] $\mathrm{Ann}_M(K) := \bigcap\{Ker(f) \mid f \in \mathrm{Hom}(M, K)\}$ is the **annihilator of K in M**; is a fully invariant submodule of M, and is the largest submodule of M such that $(K)_M K = 0$.

Theorem ([MMSZ20])

Let M be a self-progenerator in $\sigma[M]$. Assume $\Lambda^{fi}(M)$ is compact. The following conditions are equivalent:

(a) *M is strongly harmonic.*

(b) *$\Lambda^{fi}(M)$ is a normal idiom.*

(c) *For each $N \in \Lambda^{fi}(M)$ and $\mathcal{M} \in Max^{fi}(M)$*
$$Ler(N) \leq \mathcal{M} \Leftrightarrow N \leq \mathcal{M}.$$

(d) *Ler is $\sum$-preserving (Ler has right adjoin*[21]

(e) *For each $N, L \in \Lambda^{fi}(M)$*
$$N + L = M \Rightarrow Ler(N) + Ler(L) = M.$$

[21] Ler is given by, $Ler(N) = \sum\{K \in \Lambda^{fi}(M) \mid N + Ann_M(K) = M\}$, for each $N \in \Lambda^{fi}(M)$.

Theorem ([MMSZ20])

Let M be projective in $\sigma[M]$ such that $\Lambda^{fi}(M)$ is compact. Consider the following conditions:

(a) *M is a strongly harmonic module.*

(b) *$\Lambda^{fi}(M)$ is normal.*

(c) *$\Lambda^{fi}(M)_\mu$ is a normal lattice.*

(d) *$Spec(M)$ is a normal space.*

Then the implications (a)⇒(b)⇒(c)⇔(d) hold. If in addition $0 = \bigcap Max^{fi}(M)$, then the four conditions are equivalent.

Gelfand Modules

Definition ([MMSZ20])

A module M is Gelfand if for every distinct elements $N, L \in Max(M)$ there exist $N', L' \in \Lambda^{fi}(M)$ such that $L' \nleq L$, $N' \nleq N$ and $L'_M N' = 0$.

Definition ([MMSZ16])

An R-module M it is said to be a pm-module if every prime submodule is contained in a unique maximal submodule.

Remark

- *Let M be a Gelfand module and $P \leq M$ be a prime submodule. If there exist $L, N \in Max(M)$ such that $P \leq N$ and $P \leq L$ then $N = L$.*
- *Let M be a Gelfand module. Then M is a quasi-duo module (i.e $Max(M) \subseteq \Lambda^{fi}(M)$).*
- *Let M be projective in $\sigma[M]$. If M is a Gelfand module then $\Lambda^{fi}(M)$ is coatomic.*
- *Let M be a quasi-projective Gelfand module and $N \leq M$. Then M/N is a Gelfand module.*
- *In contrast to strongly harmonic modules, an arbitrary coproduct of copies of a Gelfand module might not be Gelfand.*

In [Sun91] is extended the Demarco-Orsati-Simmons Theorem for symmetric rings (which includes the commutative rings).
We could not find a good generalization of symmetric rings for modules which be suitable to give a version of the Demarco-Orsati-Simmons Theorem in the module-theoretic context.
We finish with a Theorem inspired in the Demarco-Orsati-Simmons Theorem as a compendium of our results.

Theorem ([MMSZ20])

Let M be projective in $\sigma[M]$ such that $\Lambda^{fi}(M)$ is compact and $Max(M)$ compact. Consider the following conditions

(a) *M is a Gelfand module.*
(b) *M is a quasi-duo strongly harmonic module.*
(c) *M is a quasi-duo pm-module with $Max(M)$ Hausdorff.*
(d) *M is a quasi-duo with $Max(M)$ is Hausdorff and $Max(M)$ is a retract of $Spec(M)$.*
(e) *M is a quasi-duo modulo such that $Spec(M)$ is normal.*

Then the implications (a)⇔(b)⇒(c)⇒(d)⇒(e) hold. If in addition $0 = \bigcap Max(M)$, all the conditions are equivalent.

References

[CRT18] J. Castro, J. Rios, and G. Tapiaz, **Some aspects of Zariski topology for multiplication modules and their attached frames and quantales**, Journal of the Korean Mathematical Society 56 (2018), no. 5, 1285â1307.

[MSZ15] Medina M., Sandoval L., Zaldivar A. **A generalization of quantales with applications to modules and rings.** Journal of Pure and Applied Algebra, Vol. 220, No. 5, 1837 - 1857 (2015).

[MMSZ16] Medina M., Morales L., Sandoval L., Zaldivar A. **Attaching topological spaces to a module (I): Sobriety and spatial frames of submodules.** Journal of Pure and Applied Algebra, Volume 222, Issue 5, 2018, Pages 1026-1048,

[MMSZ20] Medina M., Morales L., Sandoval L., Zaldivar A. **On Strongly harmonic and Gelfand modules.** Communications in Algebra, (2020) https://doi.org/10.1080/00927872.2019.1710167

[MSZ20] Medina M., Sandoval L., Zaldivar A. **On the De Morgan's laws for modules.** Appl Categor Struct (2021). https://doi.org/10.1007/s10485-021-09656-8

[R90] Rosenthal K.I. **Quantales and their applications.** Pitman Research Notes in Mathematics Series, Longman Higher Education (1990).

[S06] Simmons H. **The point space of a frame,** `www.cs.man.ac.uk/?hsimmons/FRAMES/C-POINTSPACE.pdf` (2006).

[Sun91] S.H. Sun, **Noncommutative rings in which every prime ideal is contained in a unique maximal ideal**, Journal of Pure and Applied Algebra 76 (1991), no. 2, 179â192.


thank you
danke
gracias
merci
obrigado
dziękuję
bedankt
teşekkür ederim
ngiyabonga
謝謝
go raibh maith agat
mochchakkeram
sukriya
kop khun krap
terima kasih
arigatô
dakujem
grazie
takk
sagolun
mauruuru
hvala
tapadh leat
спасибо
Баярлалаа
рахмат
мерси
хвала
dank je
ευχαριστώ
감사합니다
ありがとう
xiexie
mersi
barka
spas
tack
blagodaram
misaotra
malondo
paldies
mahalo
asante
manana
murakoze
tenki
djiere dieuf
mamnun
trugarez
shukriya
dhanyavadagalu
diolch
tanemirt
rahmet
najis tuke
didi madloba
mesi
dekuji
sobodi
enkosi
nanni
nandri
kiitos
dankie
koszonom
bayarlalaa
gracie
faafetai lava
vinaka
chnorakaloutioun
gracies
sulpay
taiku
akun dankon
kam sah hamnida
rahmat